



Essential curves in handlebodies and topological contractions

Viatcheslav Grines, François Laudenbach

► To cite this version:

Viatcheslav Grines, François Laudenbach. Essential curves in handlebodies and topological contractions. *Geometry and Topology*, 2008, 12, pp.981-986. hal-00176103

HAL Id: hal-00176103

<https://hal.science/hal-00176103>

Submitted on 2 Oct 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ESSENTIAL CURVES IN HANDLEBODIES AND TOPOLOGICAL CONTRACTIONS

V. GRINES AND F. LAUDENBACH

ABSTRACT. If X is a compact set, a *topological contraction* is a self-embedding f such that the intersection of the successive images $f^k(X)$, $k > 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus ≥ 2 whose image is essential. Our proof is based on an easy criterion for a simple curve to be essential in a handlebody.

1. INTRODUCTION

For a compact set X and a topological embedding $f : X \rightarrow X$, we shall say that f is a *topological contraction* if $\bigcap_{k \geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when X is a 3-dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A. *There exists a North-South diffeomorphism f of the 3-sphere S^3 and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:*

- 1) $f|_{P_+}$ is a topological contraction;
- 2) $f(P_+)$ is essential in P_+ .

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3-dimensional *handlebody* of genus 2 is diffeomorphic to the regular neighborhood P in \mathbb{R}^3 of the planar figure eight Γ . A *compression disk* of P is a smooth embedded disk in P whose boundary lies in ∂P in which it is not homotopic to a point. Among the compression disks are the *meridian* disks $\pi^{-1}(x)$, where x is a regular point¹ in Γ and $\pi : P \rightarrow \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of Γ). A subset X of P_+ is said to be *essential* in P_+ if it intersects every compression disk².

A diffeomorphism f of S^3 is a North-South diffeomorphism if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from α to ω .

A *Heegaard splitting* of S^3 is made of an embedded surface dividing S^3 into two handlebodies. According to a famous theorem of F. Waldhausen such a decomposition is unique up to diffeomorphism [4] (hence up to isotopy after Cerf's theorem $\pi_0(Diff_+ S^3) = 0$ [1]). It is not

1991 *Mathematics Subject Classification.* 57M25, 37D15.

Key words and phrases. Heegaard splitting, compression disk, North-South diffeomorphism.

¹Any point other than the center of the figure eight.

²This definition goes back to Rolfsen's book [3] p. 110.

hard to prove that the phenomenon mentioned in theorem A does not happen with a Heegaard splitting of genus 1: if T is a solid torus and f is a topological contraction of T , then there is a compression disk of T avoiding $f(T)$.

The example which we are going to construct for proving theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains P and which is a tubular neighborhood of Γ_0 . Let $i_0 : P \rightarrow T$ be this inclusion. We say that a simple curve is unknotted in T if it bounds an embedded disk in T .

Theorem B. *There exists an essential simple curve C in P such that $i_0(C)$ is unknotted in T .*

Theorem B looks very easy as it is simple to draw a simple curve which intuitively satisfies its conclusion. Nevertheless, it appears that there are very few criteria for proving that a curve is essential in P . We are going to give one which is not algebraic in nature. Question: does there exist a topological algebraic tool which plays the same role.

The second author is grateful to Sylvain Gervais and Nathan Habegger for interesting conversations on link invariants, in particular on Milnor's invariants [2]. He is also grateful to the organizers³ of the conference in Toulouse in memory of Heiner Zieschang (*Braids, groups and manifolds*, Sept. 2007), who offered him the opportunity to give a short talk on that subject.

2. ESSENTIAL CURVES

Our candidate for C in Theorem B is pictured in figure 1.

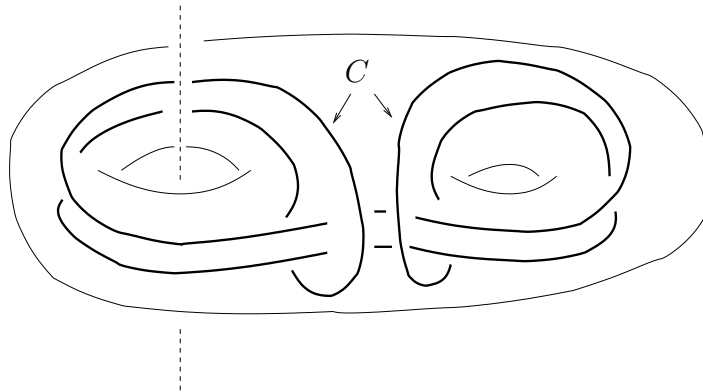


Figure 1

It is clear that $i_0(C)$ is unknotted in T (or, equivalently, in the complement of the vertical axis which is drawn on figure 1 and whose T is a compact retract by isotopy deformation). Instead of proving that C is essential in P , we are going to prove a stronger result. Clearly Proposition 1 below implies Theorem B.

³Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat.

Proposition 1. *Let $p : \tilde{P} \rightarrow P$ be the universal cover of P and let \tilde{C} be the preimage $p^{-1}(C)$. Then \tilde{C} is essential in \tilde{P} .*

Proof. We have the following description of \tilde{P} : it is a 3-ball with a Cantor set E removed from its bounding 2-sphere⁴. This Cantor set is the set of ends of \tilde{P} . A simple curve in $\partial\tilde{P}$ is not homotopic to zero if it divides E into two non-empty parts. We get a fundamental domain F for the action of $\pi_1(P)$ on \tilde{P} by cutting P along two non-parallel meridian disks D_0 and D_1 . Here is a description of $\tilde{C} \cap F$ (see figure 2): F is a 3-ball whose boundary consists of four disks d_0, d'_0, d_1, d'_1 and a punctured sphere $\partial_0 F$. We have $p(d_0) = p(d'_0) = D_0$ and $p(d_1) = p(d'_1) = D_1$. We have four strands in $\tilde{C} \cap F$: ℓ_1 and ℓ_2 joining d_0 and d_1 , ℓ'_0 (resp. ℓ'_1) whose end points belong to d'_0 (resp. d'_1). Moreover ℓ'_i , $i = 0, 1$, is linked with ℓ_j , $j = 1, 2$, in the following sense: any embedded surface whose boundary is made of ℓ'_i and a simple arc in d'_i intersects ℓ_j for $j = 1, 2$ (the algebraic intersection number is 1 for some choice of orientations).

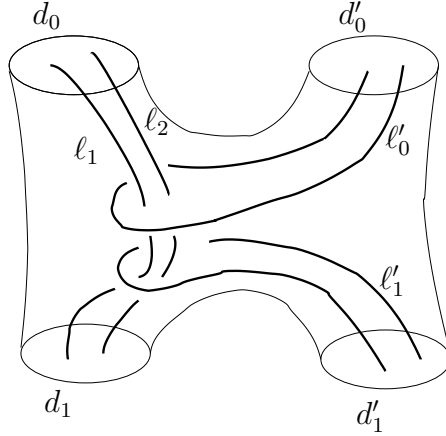


Figure 2

Globally \tilde{C} looks like an infinite Borromean chain: any finite number of components is unlinked. Suppose the contrary that \tilde{C} is not essential and consider Δ , a compression disk of \tilde{P} avoiding \tilde{C} . We take it to be transversal to $\tilde{D} := p^{-1}(D_0 \cup D_1)$. Let \mathcal{C} be the finite family of curves (arcs or closed curves) in $\tilde{D} \cap \Delta$. An element γ of \mathcal{C} is said to be *innermost* if γ divides Δ into two domains, one of them being a disk δ whose interior contains no element of \mathcal{C} . Take such an innermost element γ ; its associated disk δ lies in F , up to a covering transformation, and divides F into two balls F_0 and F_1 .

Lemma 1. *One of the balls, say F_0 , avoids \tilde{C} .*

Proof. Let us consider the case when $\gamma \subset d'_0$; say that $\ell'_0 \subset F_1$. The other cases are very similar. Let $\alpha = \delta \cap d'_0$. It is a simple arc dividing d'_0 into two parts. Both end points of ℓ'_0 lie in the same part since δ avoids ℓ'_0 . They are joined by a simple arc α' disjoint from α . Let δ' be an embedded disk bounded by $\ell'_0 \cup \alpha'$. This disk can be chosen disjoint from δ . Indeed, if

⁴Take the universal cover of Γ properly embedded in the hyperbolic plane and take a 3-dimensional thickening of it.

$\delta \cap \delta'$ is not empty, this intersection being transversal, by looking at an innermost intersection curve on δ one finds an embedded 2-sphere S in the complement of \tilde{C} with one hemisphere in δ and the other in δ' . As S bounds a 3-ball B_F in $\text{int } F$, which hence is also disjoint from \tilde{C} , there is an isotopy supported in a neighborhood of B_F whose effect on δ' decreases the number of intersection curves with δ .

Once $\delta \cap \delta'$ is empty, we have $\delta' \subset F_1$. But ℓ_1 and ℓ_2 must intersect δ' . Hence we have $\ell_1 \cup \ell_2 \subset F_1$. Similarly, we have $\ell'_1 \subset F_1$. \square

One checks easily that there is an isotopy of Δ , supported in a neighborhood of F_0 , till a new compression disk having fewer intersection curves with \tilde{D} than the cardinality of \mathcal{C} . Repeating this process, we push Δ into a fundamental domain, say F . In that position we have $\partial\Delta \subset \partial_0 F$. Again Δ divides F into two balls and one of them, F_0 , avoids \tilde{C} . This proves that $\partial\Delta$ bounds a disk in $\partial_0 F$, namely $F_0 \cap \partial_0 F$. Hence Δ is not a compression disk. \square

Remark. We used local linking information (namely, linking of strands in a fundamental domain of the universal covering space) which, as in this example, follows from usual linking numbers and we got a global result. This method looks very efficient. The general criterion is the following, where we use the same notation as above.

Criterion. *Let C be any simple closed curve in P . We assume that there is no embedded disk δ in F satisfying:*

- 1) *the boundary of δ is made of two arcs α and β , where α is an arc in \tilde{D} and β is an arc in $\partial\tilde{P} \cap F$;*
 - 2) *δ non trivially separates the components of $\tilde{C} \cap F$ (both components of $F \setminus \delta$ meet \tilde{C}).*
- Then C is essential in P .*

3. PROOF OF THEOREM A

We recall the embedding $i_0 : P \rightarrow \text{int } T$. We start with a curve C in P which meets the conclusion of Theorem B. We equip it with its 0-normal framing (a section in this framing is not linked with C in \mathbb{R}^3) and we choose an embedding $j_0 : T \rightarrow P$ whose image is a tubular neighborhood of C . Let B be a small ball in $\text{int } T$. As C is unknotted in T , there is an ambient isotopy, supported in $\text{int } T$, deforming i_0 to $i_1 : P \rightarrow \text{int } T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in B . One half of the desired Heegaard splitting of genus 2 will be given by $P_+ := i_1(P)$. At the present time f is only defined on T by $f := i_1 \circ j_0 : T \rightarrow \text{int } T$. If we compose i_1 with a sufficiently strong contraction of B into itself, then f is a contraction in the metric sense. Hence $\cap_{k>0} f^k(T)$ consists of one point.

Choose a round ball B' containing T in its interior. Since $f|_T$ is isotopic to the inclusion $T \hookrightarrow \mathbb{R}^3$, f extends as a diffeomorphism $B' \rightarrow B$, and further as a diffeomorphism $S^3 \rightarrow S^3$. We are free to choose $f : S^3 \setminus B' \rightarrow S^3 \setminus B$ as we like. If we compose f^{-1} with a strong contraction of $S^3 \setminus B'$, the intersection $\cap_k f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North-South diffeomorphism f of S^3 which induces a topological contraction of T . Since

$f(T) \subset \text{int } P_+ \subset P_+ \subset \text{int } T$, f also induces a topological contraction of P_+ .

It remains to prove that $f(P_+)$ is essential in P_+ . We know that $i_1(C)$ is essential in P_+ . As a consequence, any compression disk Δ of P_+ crosses $f(T)$. We can take Δ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let γ be an intersection curve which is *innermost* in Δ and let δ be the disk that γ bounds in Δ .

Lemma 2. *We have $\delta \subset f(T)$.*

Proof. If not, we have $\delta \subset P_+ \setminus f(\text{int } T)$ and the simple curve γ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the 0-framing. In that case, $i_1(C)$ itself bounds an embedded disk in P_+ . This is impossible, as $i_1(C)$ is essential in P_+ . \square

Therefore δ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like P itself, is essential in T . Hence $f(P_+)$ is essential in $f(T)$ and δ must cross $f(P_+)$. \square

REFERENCES

- [1] J. Cerf, Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$) Lect. Notes in Math. 53, Springer, 1968.
- [2] J. Milnor, *Isotopy of links. Algebraic geometry and topology* in: A symposium in honor of S. Lefschetz, Princeton University Press, 1957, p. 280–306.
- [3] D. Rolfsen, *Knots and Links*, Publish or Perish, Inc., Berkeley, 1976.
- [4] F. Waldhausen, *Heegaard-Zerlegungen der 3-Sphäre*, Topology 7 (1968), 195–203.

N. NOVGOROD STATE UNIVERSITY, GAGARINA 23, N. NOVGOROD, 603950 RUSSIA.
E-mail address: grines@vmk.unn.ru

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UMR 6629 DU CNRS, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ DE NANTES, 2, RUE DE LA HOUSSINIÈRE, F-44322 NANTES CEDEX 3, FRANCE.
E-mail address: francois.laudenbach@univ-nantes.fr